FIRST ORDER S OPERATOR TERMS IN QED

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We would like to apply the S operator to the interaction term in quantum electrodynamics (QED). The full Lagrangian for a system consisting of electrons and photons, including their interaction is

$$\mathcal{L}^{1/2,1} = -\frac{1}{2}\left(\partial_\nu A_\mu\right)\left(\partial^\nu A^\mu\right) + \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + e\bar{\psi}\gamma^\mu A_\mu\psi \tag{1}$$

The first two terms represent the free photon field and free electron field, respectively. The last term is the interaction. The interaction Hamiltonian is therefore

$$\mathcal{H}_I^I = -\mathcal{L}_I^I = -e\bar{\psi}\gamma^\mu A_\mu\psi \tag{2}$$

Using the slash notation to indicate a sum with the gamma matrices, we have

$$\mathcal{H}_I^I = -e\bar{\psi}\not{A}\psi \tag{3}$$

We now wish to use this Hamiltonian density to evaluate the S operator up to first order. That is, we seek e^-

$$S_{oper}\left(t, t_i\right) = T\exp\left[-i\int_{t_i}^{t}\mathcal{H}_I^I d^4x\right] \tag{4}$$

$$= I - i\int\mathcal{H}_I^I d^4x + \ldots \tag{5}$$

where T indicates time-ordering. We can define the first order term as

$$S^{(1)} \equiv -iT\int\mathcal{H}_I^I d^4x \tag{6}$$

To do this, we need the explicit forms of the various field operators. For A^μ, we have

$$A^{\mu}(x) = \sum_{s,\boldsymbol{k}} \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}} \varepsilon_s^{\mu} a_s(\boldsymbol{k}) e^{-ikx} + \sum_{s,\boldsymbol{k}} \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}}}} \varepsilon_s^{\mu} a_s^{\dagger}(\boldsymbol{k}) e^{ikx} \tag{7}$$

$$= A^{\mu+} + A^{\mu-} \tag{8}$$

The subscript s indicates which of the 4 polarization vectors ε_s^{μ} we're considering, and the superscript μ indicates which of the 4 components of the 4-vector we're dealing with.

For electrons, we have the solution ψ and its adjoint $\bar{\psi}$:

$$\psi = \sum_{r=1}^{2} \sum_{\mathbf{p}} \sqrt{\frac{m}{VE_{\mathbf{p}}}} \left[c_r(\mathbf{p}) u_r(\mathbf{p}) e^{-ipx} + d_r^{\dagger}(\mathbf{p}) v_r(\mathbf{p}) e^{ipx} \right] \tag{9}$$

$$\equiv \psi^{+} + \psi^{-} \tag{10}$$

$$\bar{\psi} = \sum_{r=1}^{2} \sum_{\mathbf{p}} \sqrt{\frac{m}{VE_{\mathbf{p}}}} \left[d_r(\mathbf{p}) \bar{v}_r(\mathbf{p}) e^{-ipx} + c_r^{\dagger}(\mathbf{p}) \bar{u}_r(\mathbf{p}) e^{ipx} \right] \tag{11}$$

$$\equiv \bar{\psi}^{+} + \bar{\psi}^{-} \tag{12}$$

The first order term 6 is obtained by substituting these explicit solutions into 2 and integrating. That is

$$S^{(1)} = ieT \int d^4x \left(\bar{\psi}^{+} + \bar{\psi}^{-}\right) \left(\not{A}^{\mu+} + \not{A}^{\mu-}\right) \left(\psi^{+} + \psi^{-}\right) \tag{13}$$

When multiplied out, we get a sum of 8 terms. In first order, all the time variables in the various field operators are equal at each stage in the integration, so the time-ordering is redundant. From Wick's theorem, the conversion from time-ordered to normal-ordered for two operators is

$$T[BC] = N[BC] + \underbrace{BC}_{\sqcup} \tag{14}$$

However, we can use the fact that the contraction of two operators at equal times is zero (to be confirmed in a later post), so that

$$T[BC] = N[BC] \tag{15}$$

When we apply this to each term in 13, we recall that terms with a + superscript are destruction operators and terms with a $-$ superscript are creation operators, so we must put all + terms on the right when normal ordering. The A^{μ} terms destroy and create photons, ψ destroys electrons via c_r and creates positrons via $d_r^{\dagger}$, and $\bar{\psi}$ destroys positrons via d_r and creates electrons via $c_r^{\dagger}$.

Klauber goes through a detailed example of the term $\bar{\psi}^{+}A_{\mu}^{-}\gamma^{\mu}\psi^{+}$ acting on an initial state containing two electrons. Here, we'll go through the calculation of the term $\bar{\psi}^{-}\not{A}^{+}\psi^{+}$, which destroys an initial photon and electron (via $\not{A}^{+}\psi^{+}$) and creates another electron via $\bar{\psi}^{-}$. Thus the process models the absorption of a photon by an electron (which turns out to be physically impossible, but more on that later).

The term $\bar{\psi}^{-}\not{A}^{+}\psi^{+}$ is already normal ordered, since all the destruction operators are on the right. We suppose that this term acts on an initial state

$$|i\rangle = \left|\gamma_{\boldsymbol{k}_1,s_1}, e^{-}_{\boldsymbol{p}_1,r_1}\right\rangle \tag{16}$$

In this case, all photon destruction operators other than the one with $\boldsymbol{k} = \boldsymbol{k}_1$ and $s = s_1$ will produce 0, since the corresponding photon does not exist in the initial state. Similarly, all electron destruction operators other than the one with $\boldsymbol{p} = \boldsymbol{p}_1$ and $r = r_1$ will produce 0. The remaining two destruction operators will remove the photon and electron, leaving the vacuum state. Therefore

$$\bar{\psi}^{-}\not{A}^{+}\psi^{+}\left|\gamma_{\boldsymbol{k}_1,s_1}, e^{-}_{\boldsymbol{p}_1,r_1}\right\rangle = ie\int d^4x\left[\sum_{r=1}^{2}\sum_{\mathbf{p}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}c_r^{\dagger}(\mathbf{p})\,\bar{u}_r(\mathbf{p})\,e^{ipx}\right]\times \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}_1}}}\varepsilon_{s_1}^{\mu}\gamma_{\mu}e^{-ik_1x}\sqrt{\frac{m}{VE_{\mathbf{p}_1}}}u_r(\mathbf{p}_1)\,e^{-ip_1x}\,|0\rangle \tag{17}$$

The only place that x occurs is in the exponentials, so we can do the integral to get

$$\int d^4x\, e^{i(p-k_1-p_1)x} = (2\pi)^4\,\delta^{(4)}(p-k_1-p_1) \tag{18}$$

The electron creation operator $c_r^{\dagger}(\mathbf{p})$ will then create an electron with momentum $\boldsymbol{p}$ and spin r. The sum over $\boldsymbol{p}$ and r in 17 produces a superposition of states, each with an electron with a given momentum and spin. If we now measure the final state to be an electron with $\boldsymbol{p} = \boldsymbol{p}_2$ and $r = r_2$, then the only term in the sums in 17 that survive are those with $\boldsymbol{p} = \boldsymbol{p}_2$ and $r = r_2$, so we have

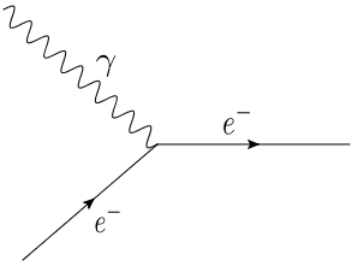

FIGURE 1. Feynman diagram for $\gamma + e^- \to e^-$.

$$\left\langle e^-_{\mathbf{p}_2,r_2} \left| \bar{\psi}^- \not{A}^+ \psi^+ \right| \gamma_{\boldsymbol{k}_1,s_1}, e^-_{\mathbf{p}_1,r_1} \right\rangle = \tag{19}$$

$$(2\pi)^4 \delta^{(4)}(p_2 - k_1 - p_1)\, ie \sqrt{\frac{m}{VE_{\mathbf{p}_2}}} \bar{u}_{r_2}(\mathbf{p}_2) \frac{1}{\sqrt{2V\omega_{\boldsymbol{k}_1}}} \varepsilon^\mu_{s_1} \gamma_\mu \sqrt{\frac{m}{VE_{\mathbf{p}_1}}} u_r(\mathbf{p}_1) \left\langle e^-_{\mathbf{p}_2,r_2} \left| e^-_{\mathbf{p}_2,r_2} \right. \right\rangle = \tag{20}$$

$$(2\pi)^4 \delta^{(4)}(p_2 - k_1 - p_1)\, ie \frac{m\bar{u}_{r_2}(\mathbf{p}_2)\, \varepsilon^\mu_{s_1} \gamma_\mu u_r(\mathbf{p}_1)}{V^{3/2}\sqrt{2E_{\mathbf{p}_1} E_{\mathbf{p}_2} \omega_{\boldsymbol{k}_1}}} \tag{21}$$

where in the last line, we used the normalization of the state so that

$$\left\langle e^-_{\mathbf{p}_2,r_2} \left| e^-_{\mathbf{p}_2,r_2} \right. \right\rangle = 1 \tag{22}$$

The delta function imposes conservation of momentum, so that we must have

$$p_2 = k_1 + p_1 \tag{23}$$

This process represents a photon being absorbed by an electron. The Feynman diagram is shown in Fig. 1, where time increases from left to right.

As mentioned above, this process is physically impossible. To see why, we can view the system in its centre of momentum frame. Before the interaction, the photon and electron are moving towards each other so the total energy in the system is equal to the energy of the photon plus the energy of the moving electron. After the interaction, in the centre of momentum frame we have a single electron at rest, so the total energy is just the rest mass of the electron. However, the total energy in the system before the interaction is greater than the electron's rest mass, so it is impossible to conserve energy.

PINGBACKS

Pingback: Electron-electron scattering in second order S matrix